United Kingdom Hydrographic Office

HM NAUTICAL ALMANAC OFFICE

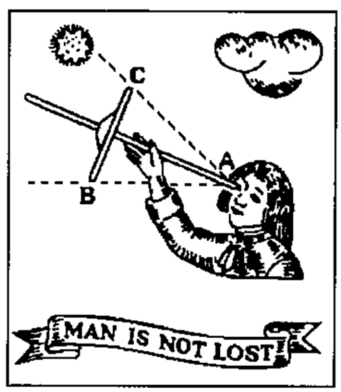
NAO TECHNICAL NOTE

No. 74 2010 August

Computation of the Quantities Describing the Lunar Libations in *The Astronomical Almanac*

By

D.B. Taylor & S.A. Bell, HM Nautical Almanac Office
J.L. Hilton, US Naval Observatory
A.T. Sinclair, HM Nautical Almanac Office, retired



HM Nautical Almanac Office
UK Hydrographic Office
Admiralty Road, Taunton
TA1 2DN England

Telephone: +44 (0) 1823 337900
Facsimile: +44 (0) 1823 335396
e-mail: hmnao@ukho.gov.uk
WWW http://www.hmnao.com/

Report Documentation Page						*Form Approved* *OMB No. 0704-0188*
Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.						
1. REPORT DATE **AUG 2010**		2. REPORT TYPE				3. DATES COVERED **00-00-2010 to 00-00-2010**
4. TITLE AND SUBTITLE **Computation of the Quantities Describing the Lunar Librations in the Astronomical Almanac**						5a. CONTRACT NUMBER
						5b. GRANT NUMBER
						5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)						5d. PROJECT NUMBER
						5e. TASK NUMBER
						5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **United Kingdom Hydrographic Office,Admiralty Road, Taunton,TA1 2DN England, ,**						8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)						10. SPONSOR/MONITOR'S ACRONYM(S)
						11. SPONSOR/MONITOR'S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release; distribution unlimited**						
13. SUPPLEMENTARY NOTES						
14. ABSTRACT						
15. SUBJECT TERMS						
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON	
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**10**		

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

Computation of the Quantities Describing the Lunar Librations in The Astronomical Almanac

D.B. Taylor and *S.A. Bell,* HM Nautical Almanac Office

J.L. Hilton, US Naval Observatory

A.T. Sinclair, HM Nautical Almanac Office, retired

Contents

Appendices

1. Introduction

Lunar physical ephemerides, tabulated daily including libration, can be found in Section D of *The Astronomical Almanac* (AsA). This technical note describes the method used for calculating the lunar librations and other quantities on the odd pages D7–D21 of the AsA with effect from the 2011 edition. Prior to the 1985 edition, formulae and constants for the physical ephemeris of the Moon were due to Hayn (1907). Beginning with the 1985 edition, librations have been generated using the analytical theory of Eckhardt (1981, 1982).

Since the inclusion of a rotational ephemeris for the Moon given in the form of three Euler angles in the JPL lunar ephemerides starting with DE403/LE403, Standish et al. (1995), it has been a long-standing goal to implement these ephemerides in the calculation of lunar librations and improve the quality of the tabulated physical ephemerides of the Moon in Section D. Work started by Hilton (2004) on interpreting these ephemerides has paved the way to the work presented here which describes the calculation of librations and related quantities as they have been implemented starting with the 2011 edition of the AsA. This technical note has been written with the practitioner of astronomical calculations in mind and may form the basis of explanatory material on lunar librations in a future edition of *The Explanatory Supplement to the Astronomical Almanac* (ES).

In this technical note the basic formulae are derived for the lunar librations and the position angle of the axis of rotation. The lunar rotation angles in the JPL ephemeris must be transformed by a series of rotations and then using simple vector algebra to angular quantities which can be substituted into the basic formulae. These will then give the true or total lunar librations. From these and the optical (geometric) librations, the physical librations can be computed. For the 2011 edition, the lunar rotation angles in DE403/LE403 are used due to the availability of certain ephemeris-specific transformations (Konopliv et al. (2001)).

An algorithm and numerical example, using the same computer routines used for the AsA, showing all the relevant stages in the computation of the librations and related quantities is given. Appendix C contains a short description of the calculation of the position angle of the Moon's bright limb and its phase or fraction illuminated – geometrical quantities tabulated on the odd pages D7–D21 of Section D.

In the ES (1961) an approximate method was given to calculate the lunar librations but with no details of its derivation. In Appendix B these formulae are derived based on the method in Encke (1843). This method was used primarily before the advent of high-speed electronic computers and is not used in the publications currently to calculate the lunar librations. It is included here for the interested reader.

2. Basic formulae for the lunar librations and the position angle of the axis

2.1 The lunar librations

The mean rotational state of the Moon is described by Cassini's empirical laws, which state that the descending node of the Moon's equator coincides with the ascending node of the Moon's orbit on the ecliptic; the Moon's equator maintains a constant inclination to the ecliptic; and the rotation rate is such that on average the same side is always facing the Earth. Thus the rotational rate must be equal to the rate of motion of the Moon's mean longitude. The actual rotation state has small periodic variations from this mean state caused by dynamical perturbations, and these cause the physical librations of the Moon's orientation. In addition there are the much larger optical librations in its orientation as seen from the Earth, which are due to variations in the rate of the Moon's orbital motion, and to the inclination of the Moon's equator to its orbital plane (see Appendix A for an estimation of the magnitude of these librations). For a complete definition of the rotational state of the Moon, a prime meridian must also be specified, and this was originally chosen to be the mean central meridian of the side facing the Earth. Its direction in space will thus differ by 180° from the mean longitude of the Moon. The situation is illustrated in Figure 1.

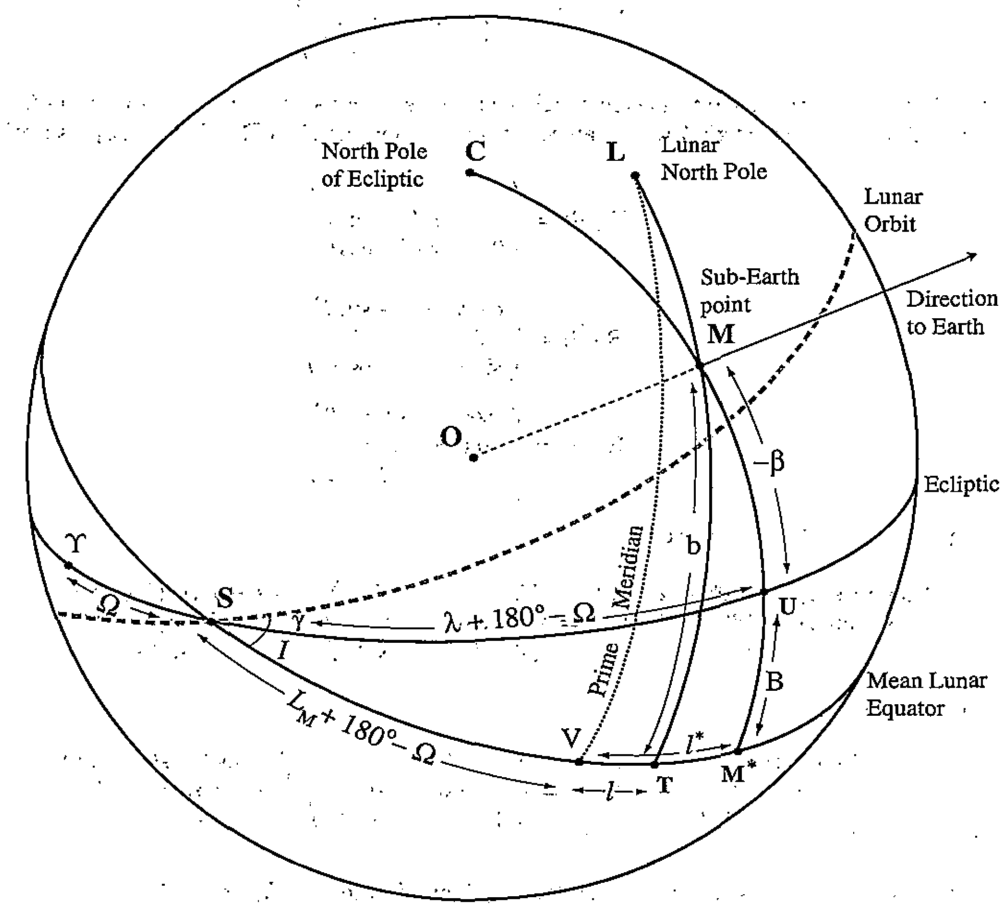


Figure 1: The selenocentric sphere: showing the lunar orbit and the relationships between the sub-Earth point M, *the mean lunar equator and the ecliptic.* S *is the descending node of the lunar equator on the ecliptic.*

The ecliptic longitude and latitude of the Moon are λ, β respectively and so the sub-Earth point M has longitude and latitude $\lambda + 180°$, $-\beta$. L_M is the mean longitude of the Moon and Ω is the ascending node. The inclination of the ecliptic to the mean lunar equator is I. The librations in longitude and latitude are denoted by l and b respectively.

Formulae for computing the optical librations can be derived by relating to each other two expressions for the vector from the centre of the Moon towards the sub-Earth point M; one of them referred to the ecliptic frame and the other to the lunar equatorial frame. Let O be the centre of the selenocentric sphere. In Figure 1 consider firstly point M referred to right-handed axes O_{xyz} in which O_x is in direction OS, O_y in the ecliptic and O_z in direction OC. Coordinates of M in this system are

$$\begin{pmatrix} \cos(-\beta)\cos(\lambda+180^\circ-\Omega) \\ \cos(-\beta)\sin(\lambda+180^\circ-\Omega) \\ \sin(-\beta) \end{pmatrix} \tag{1}$$

Consider next, point M referred to axes $O_{x'y'z'}$ in which $O_{x'}$ is in direction OS, $O_{y'}$ in the mean equator of the Moon and $O_{z'}$ in the direction OL. The coordinates of M referred to these axes are

$$\begin{pmatrix} \cos b\cos(L_M+180^\circ-\Omega+l) \\ \cos b\sin(L_M+180^\circ-\Omega+l) \\ \sin b \end{pmatrix} \tag{2}$$

The vectors in equations (1) and (2) are related through the rotation matrix

$$\mathbf{R}_1(-I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix} \tag{3}$$

We note here in general that rotations in a right-handed set of axes with origin O about the O_x,O_y and O_z axes through an arbitrary angle θ are obtained by the rotation matrices $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ respectively, where

$$\mathbf{R}_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{4}$$

$$\mathbf{R}_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{5}$$

$$\mathbf{R}_3(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

From equations (1), (2) and (3) we have

$$\begin{pmatrix} \cos b\cos(L_M+180^\circ-\Omega+l) \\ \cos b\sin(L_M+180^\circ-\Omega+l) \\ \sin b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix} \begin{pmatrix} \cos(-\beta)\cos(\lambda+180^\circ-\Omega) \\ \cos(-\beta)\sin(\lambda+180^\circ-\Omega) \\ \sin(-\beta) \end{pmatrix}$$

which can be written as

$$\begin{pmatrix} -\cos b\cos(L_M+l-\Omega) \\ -\cos b\sin(L_M+l-\Omega) \\ \sin b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix} \begin{pmatrix} -\cos\beta\cos(\lambda-\Omega) \\ -\cos\beta\sin(\lambda-\Omega) \\ -\sin\beta \end{pmatrix} \tag{7}$$

In the lunar ephemeris λ is referred to the true equinox of date, but L_M and Ω are referred to the mean equinox. The quantity $\lambda-\Omega$ in equation (7) must be replaced by $\lambda-(\Omega+N)$, where N is the nutation in longitude. We can now write (7) as the three equations

$$\cos b\cos(l+L_M-\Omega) = \cos\beta\cos(\lambda-\Omega-N) \tag{8}$$

$$\cos b\sin(l+L_M-\Omega) = \cos I\cos\beta\sin(\lambda-\Omega-N) - \sin I\sin\beta \tag{9}$$

$$\sin b = -\sin I\cos\beta\sin(\lambda-\Omega-N) - \cos I\sin\beta. \tag{10}$$

Equations (8), (9) and (10) are rigorous formulae for the computation of the optical librations l and b, from the values of I, Ω and L_M, which describe the mean rotational state of the Moon, and from λ and β, the ecliptic coordinates of the Moon. However, as is explained in Section 3, if I, Ω and L_M are substituted by modified quantities that include the effects of the dynamical perturbations of the Moon's rotation, then these rigorous formulae will give the values of the total librations.

An approximate method to compute the librations l, b was given in the ES (1961) p.319 based on formulae introduced by Encke (1843). It is derived using the quantities B and l^* shown in Figure 1 and was used before the advent of fast electronic computers. A statement of the method and an outline of its derivation is given in Appendix B for the interested reader.

2.2 The position angle of the axis

The position angle of the axis of rotation is the angle that the lunar meridian through the apparent central point of the disk towards the north lunar pole forms with the celestial meridian through the central point, measured eastwards from the celestial north point of the disk.

In determining expressions for the position angle we use the elements of the mean lunar equator referred to the Earth equator.

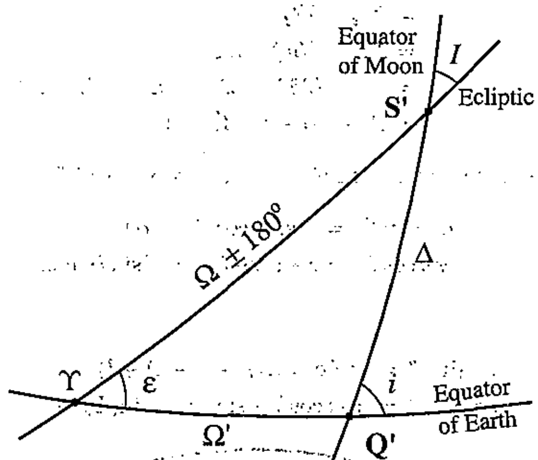


Figure 2: Elements for Moon's equator.

These are defined as:

i = the inclination of the mean equator of the Moon to the true equator of the Earth;

Δ = the arc of the mean equator of the Moon from its ascending node on the true equator of the Earth to its ascending node on the ecliptic of date;

Ω' = the arc of the true equator of the Earth from the true equinox of date to the ascending node of the mean equator of the Moon on the true equator of the Earth.

The ascending node of the mean lunar equator on the ecliptic is at the descending node of the mean lunar orbit so $\Upsilon S' = \Omega \pm 180°$. ε is the true obliquity and the node is referred to the true equinox by increasing Ω by the nutation in longitude N. From the spherical triangle $\Upsilon S'Q'$ in Figure 2 the elements can be found from the formulae

$$\sin\Delta\sin i = -\sin\varepsilon\sin(\Omega + N) \tag{11}$$

$$\cos\Delta\sin i = \sin I\cos\varepsilon - \cos I\sin\varepsilon\cos(\Omega + N) \tag{12}$$

$$\cos i = \cos I\cos\varepsilon + \sin I\sin\varepsilon\cos(\Omega + N) \tag{13}$$

$$\sin\Omega'\sin i = -\sin I\sin(\Omega + N) \tag{14}$$

$$\cos\Omega'\sin i = \cos I\sin\varepsilon - \sin I\cos\varepsilon\cos(\Omega + N). \tag{15}$$

In Figure 3 the position angle C' of the axis is shown on the selenocentric sphere. The geocentric right ascension and declination of the Moon are α, δ and so the right ascension and declination of the sub-Earth point are $\alpha + 180°$, $-\delta$. The descending node of the lunar equator on the ecliptic is denoted by point S. From the definition of Δ and Ω' the arcs $SQ = 360° - \Delta$ and $\Upsilon Q = \Omega' + 180°$. In the spherical triangle

NLM all angles and sides are known except angles $L\hat{N}M$ and $M\hat{L}N$. Noting that $XQ = 90°$ and $YQ = 90°$ these are found as follows:

$$\begin{aligned} L\hat{N}M &= 180° - YP \\ &= 180° - (\Upsilon P - \Upsilon Y) \\ &= 180° - \Upsilon P + (\Upsilon Q - YQ) \\ &= 180° - (\alpha + 180°) + (\Omega' + 180°) - 90° \\ &= \Omega' - \alpha + 90° \end{aligned} \tag{16}$$

$$\begin{aligned} M\hat{L}N &= XT \\ &= XQ - TQ \\ &= XQ - (SQ - ST) \\ &= 90° - (360° - \Delta) + (L_M - \Omega + 180° + l) \\ &= \Delta + L_M + l - \Omega - 90°. \end{aligned} \tag{17}$$

The position angle (C') can be found from either of the two sets of formulae

$$\cos b \sin C' = -\sin i \cos(\Omega' - \alpha) \tag{18}$$

$$\cos b \cos C' = \cos\delta \cos i - \sin\delta \sin i \sin(\Omega' - \alpha) \tag{19}$$

or

$$\cos\delta \sin C' = \sin i \cos(L_M - \Omega + \Delta + l) \tag{20}$$

$$\cos\delta \cos C' = \cos i \cos b - \sin i \sin b \sin(L_M - \Omega + \Delta + l). \tag{21}$$

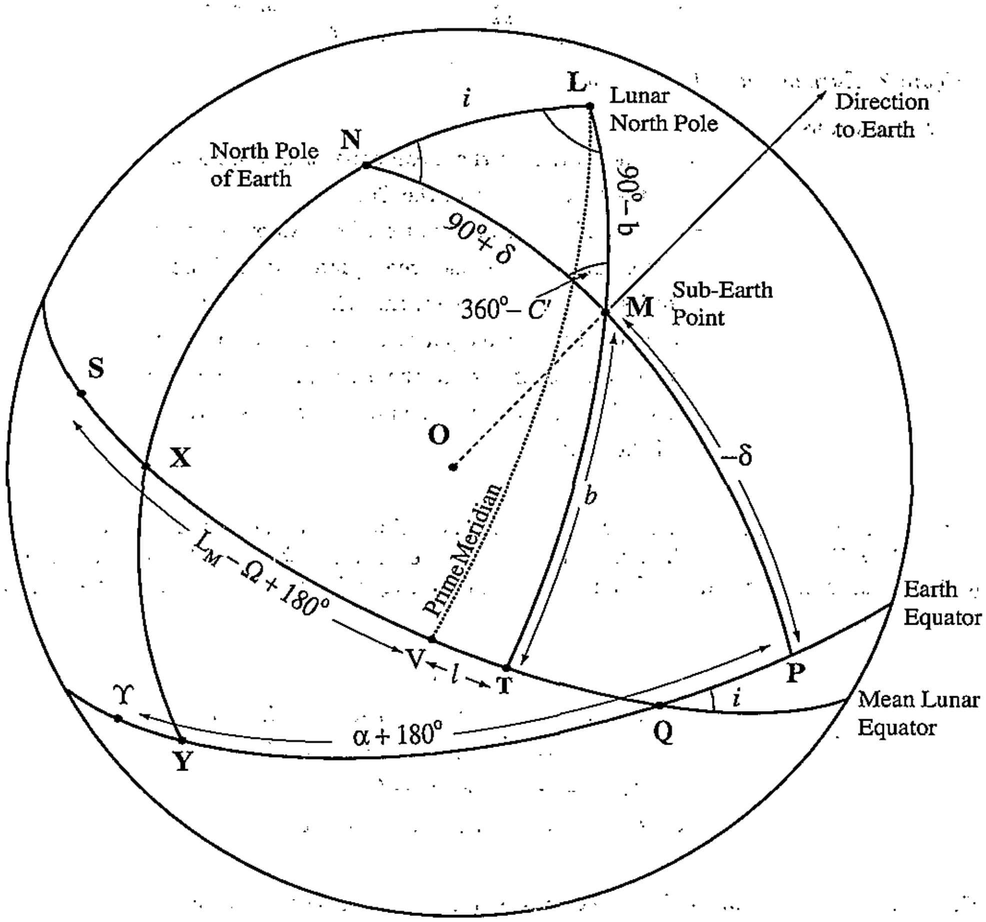


Figure 3: The selenocentric sphere: showing the relationships between the sub-Earth point M, *the mean lunar equator and the Earth's equator.*

3. Implementation of JPL lunar rotation angles

The JPL DE403/LE403 ephemeris, commonly known as DE403, includes an ephemeris for the rotation of the Moon. The problem is to transform these rotation angles into variables equivalent to the quantities Ω, I, L_M and hence use equations (8), (9), (10), and (18), (19) (or (20), (21)) with these newly determined quantities to compute librations l_T, b_T and position angle C'_T. We refer to l_T, b_T as the true or total lunar librations.

The Euler angles describing the rotation of the Moon, ϕ, θ and ψ are determined in Newhall and Williams (1997). These angles are defined relative to the ICRS Earth equator and equinox. They describe the orientation of the principal axes of inertia of the Moon (the PA system), sometimes called the axes of figure system. We show how they can be transformed to give new Euler angles ϕ_C, θ_C and ψ_C, which are defined relative to the ecliptic and equinox reference frame of date. These transformed angles are used to describe the orientation of a slightly different lunar axis system, which has one axis towards the mean Earth direction, and another along the rotation axis (the ME system), sometimes called the mean Earth/rotation axis system. From these quantities we can compute new Ω, I and L_M.

The Euler angles ϕ, θ, ψ and new set ϕ_C, θ_C, ψ_C are shown in Figures 4a, 4b respectively. These are given in Newhall and Williams (1997) and are included in this technical note for convenience.

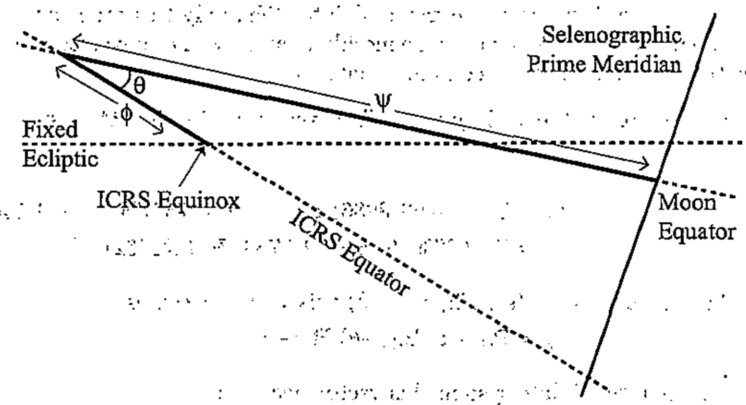


Figure 4a: Equatorial reference frame showing the Euler angles ϕ, θ, ψ, used to describe the lunar principal axis (PA) system. The value of ϕ shown in this diagram is negative.

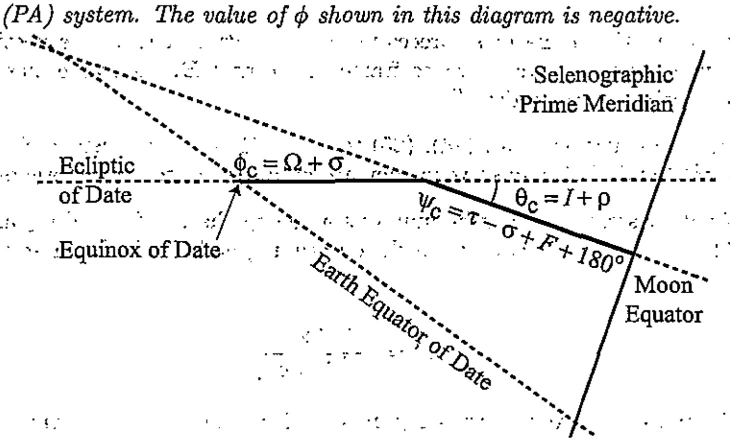


Figure 4b: Ecliptic reference frame showing the transformed Euler angles ϕ_C, θ_C, ψ_C, used to describe the orientation of the lunar ME system.

The angles are defined as:

ϕ = the angle along the ICRS equator, from the ICRS X-axis to the ascending node of the lunar equator;

θ = the inclination of the lunar equator to the ICRS equator;

ψ = the angle along the lunar equator from the node to the lunar prime meridian;

ϕ_C = the angle from the equinox of date to the descending node of the lunar equator on the ecliptic of date;

θ_C = the inclination of the lunar equator to the ecliptic of date;

ψ_C = the angle along the lunar equator from its descending node on the ecliptic to the lunar prime meridian.

From Seidelmann et al. (2007) a vector $\mathbf{p}$ in the PA system can be transformed to a vector $\mathbf{q}$ in the ME system by applying three small rotations. This expression (Konopliv et al. (2001)) is

$$\mathbf{q} = \mathbf{R}_1(-0''.1462)\,\mathbf{R}_2(-79''.0768)\,\mathbf{R}_3(-63''.8986)\,\mathbf{p} \tag{22}$$

where $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ are given in equations (4), (5), (6) respectively. It must be noted that the numerical values for the rotations in equation (22) are specific to DE403 and are different for other ephemerides.

The JPL lunar libration ephemeris gives the orientation of the PA system relative to the ICRS. So if we know the components of any vector in the PA system, then the JPL libration angles enable us to refer it to the ICRS. However the commonly-used reference system for lunar cartography is the ME system. Using the inverse of equation (22) we can convert any vector in the ME system into the PA system, and then we can use the JPL libration angles to convert to the ICRS. Finally we apply frame bias, precession and nutation, and convert to the ecliptic reference frame of date. We apply this process to vectors in the ME system that define the lunar pole and the lunar prime meridian, and use the converted vectors to calculate lunar libration angles for the ME system in the ecliptic-of-date frame.

Let $\mathbf{r}^{\rm sel}$ be a vector in the ME system. The vector $\mathbf{r}_1$ in the PA system is found from the inverse of equation (22)

$$\begin{aligned}\mathbf{r}_1 &= \mathbf{R}_3{}^{-1}(-63''.8986)\,\mathbf{R}_2{}^{-1}(-79''.0768)\,\mathbf{R}_1{}^{-1}(-0''.1462)\,\mathbf{r}^{\rm sel}\\ &= \mathbf{R}_3(63''.8986)\,\mathbf{R}_2(79''.0768)\,\mathbf{R}_1(0''.1462)\,\mathbf{r}^{\rm sel}.\end{aligned} \tag{23}$$

In the ICRS equator system (see Figure 4(a)) this vector becomes

$$\mathbf{r}_2 = \mathbf{R}_3(-\phi)\,\mathbf{R}_1(-\theta)\,\mathbf{R}_3(-\psi)\,\mathbf{r}_1. \tag{24}$$

Finally, in the ecliptic of date system this vector becomes

$$\mathbf{r}^{\rm date} = \mathbf{R}_1(\varepsilon)\,\mathbf{N}\,\mathbf{P}\,\mathbf{B}\,\mathbf{r}_2 \tag{25}$$

where $\mathbf{B}$, $\mathbf{P}$ and $\mathbf{N}$ are the frame bias, precession and nutation matrices respectively, and $\mathbf{R}_1(\varepsilon)$ is the rotation to the true ecliptic of date reference frame. Note that $\mathbf{B}$, the frame bias matrix, is included for completeness, as its effect is well under $0''.1$.

Now apply rotations in equations (23), (24), (25) to $\mathbf{r}^{\rm sel} = (1,0,0)$ and then to $\mathbf{r}^{\rm sel} = (0,0,1)$ and call the resulting vectors $\mathbf{x}^{\rm date}$ and $\mathbf{z}^{\rm date}$ respectively; they are with respect to the ecliptic of date system.

In the ecliptic of date system let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be unit vectors along the O_x, O_y and O_z axes respectively. Define the unit vector $\mathbf{\Omega}$ to be in the direction O to the descending node of the lunar equator on the ecliptic (S). We have

$$\mathbf{\Omega} = \frac{\mathbf{z}^{\rm date}\times\mathbf{k}}{|\,\mathbf{z}^{\rm date}\times\mathbf{k}\,|} \tag{26}$$

The angles ϕ_C, θ_C and ψ_C are found, using Figure 4(b), from the following formulae

$$\cos\phi_C = \mathbf{i}.\mathbf{\Omega} \tag{27}$$

$$\sin\phi_C = \mathbf{j}.\mathbf{\Omega} \tag{28}$$

$$\cos\theta_C = \mathbf{k}.\mathbf{z}^{\rm date} \tag{29}$$

$$\cos\psi_C = \mathbf{\Omega}.\mathbf{x}^{\rm date} \tag{30}$$

$$\sin\psi_C = (\mathbf{z}^{\rm date}\times\mathbf{\Omega}).\mathbf{x}^{\rm date}. \tag{31}$$

From Newhall and Williams (1997)

$$\phi_C = \Omega + \sigma \tag{32}$$

$$\theta_C = I + \rho \tag{33}$$

$$\psi_C = \tau - \sigma + F + 180^\circ. \tag{34}$$

Since (see ES (1961) p.107)

$$F = L_M - \Omega \tag{35}$$

equation (34) becomes using (35)

$$\psi_C = \tau - \sigma + L_M - \Omega + 180°. \tag{36}$$

Substituting now for Ω from equation (32) into (36) we have

$$\psi_C = \tau - \sigma + L_M - \phi_C + \sigma + 180°$$
$$\psi_C + \phi_C - 180° = L_M + \tau. \tag{37}$$

From equations (32), (33) and (37) the changes in Ω, I and L_M viz. σ, ρ and τ respectively are as a result of the physical libration. To obtain the total or true librations l_T, b_T we must therefore substitute ϕ_C, θ_C and $\psi_C + \phi_C - 180°$ for Ω, I and L_M respectively in equations (8), (9) and (10). We note since the values for Ω and L_M that are substituted into equations (8), (9) and (10) are referred to the true equinox we must set $N = 0$. The position angle C'_T for the total librations is then found using equations (18), (19) or (20), (21).

We then can calculate the physical librations δl_P, δb_P and $\delta C'_P$ from

$$\delta l_P = l_T - l_O \tag{38}$$
$$\delta b_P = b_T - b_O \tag{39}$$
$$\delta C'_P = C'_T - C'_O \tag{40}$$

where the optical librations l_O, b_O and the position angle C'_O are as computed from equations in section 2.

The heliocentric ecliptic longitude, λ_H, and latitude, β_H, of the Moon are determined by calculating the vectors for the geocentric ecliptic positions of the Sun and Moon and forming the heliocentric vector to the Moon $(X, Y, Z)_{SM}$ and its length, d_{SM}:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_M = d \begin{pmatrix} \cos\lambda\cos\beta \\ \sin\lambda\cos\beta \\ \sin\beta \end{pmatrix} \tag{41}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_S = d_S \begin{pmatrix} \cos\lambda_S\cos\beta_S \\ \sin\lambda_S\cos\beta_S \\ \sin\beta_S \end{pmatrix} \tag{42}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{SM} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_M - \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_S \tag{43}$$

$$d_{SM} = \sqrt{X_{SM}^2 + Y_{SM}^2 + Z_{SM}^2} \tag{44}$$

$$\tan\lambda_H = \frac{Y_{SM}}{X_{SM}} \tag{45}$$

$$\sin\beta_H = \frac{Z_{SM}}{d_{SM}} \tag{46}$$

By substituting λ_H for λ and β_H for β in equations (8) and (9), the selenographic longitude of the Sun, l_S, can be determined in a similar manner to the libration in longitude. Similarly, substitution of these quantities into equation (10) allows the determination of the selenographic latitude, b_S, in a similar manner to the libration in latitude. The selenographic colongitude of the Sun is simply $90° - l_S$, adjusted to lie in the range 0° to 360°.

4. Numerical Example

The purpose of this numerical example is to calculate the quantities on the odd pages D7–D21 of the AsA, most of which involve the lunar librations. The quantities tabulated are:

- The Earth's selenographic longitude (l_T) and latitude (b_T), which correspond to the total librations in longitude and latitude, respectively.
- The position angle of the axis of rotation (C'_T).
- The physical librations, δl_P, δb_P and the difference $\delta C'_P$, tabulated in thousandths of a degree.
- The Sun's selenographic colongitude ($90° - l_S$) and latitude (b_S).
- The geometrical quantities fraction illuminated (f_i) and the position angle of the bright limb (PA_B).

4.1 Algorithm for calculating the librations and position angle of the axis

The method used involves two passes through a common process. The first pass calculates the optical librations, which are geometrical in nature and adhere to Cassini's laws. The second pass calculates the total librations which includes the adopted rotational ephemeris of the Moon. In this example the JPL DE403 Ephemeris is used (see Section 3). IAU Standards of Fundamental Astronomy routines (SOFA) are used for obtaining the frame bias, precession and nutation. The steps are given below.

Step 1 Obtain the apparent positions α, δ, d, and α_s, δ_s, d_s for the Moon and Sun, respectively, at time t.

Step 2 Obtain the nutation in longitude ($\Delta\psi$) and obliquity ($\Delta\varepsilon$) and the true obliquity (ε) for time t.

Step 3 Determine the apparent ecliptic positions λ, β, and λ_s, β_s, by rotating the apparent equatorial coordinates around the X-axis by the angle ε.

Step 4 Calculate the light time correction for the Moon, $\tau = d/c$, where $c = 173{\cdot}1446\,3268\,467$ is the speed of light in au/day.

Step 5 At time $t - \tau$ form the true inclination $I_m + \Delta\varepsilon$, where $\Delta\varepsilon$ is the nutation in obliquity and I_m is the Newhall and Williams (1997) value of the inclination.

Step 6 Also at time $t - \tau$ evaluate the fundamental arguments Ω and L_M using Simon et al (1994).

Step 7 Using equations (8), (9) and (10), calculate the optical (geometric) librations l and b from $\lambda - \Omega - N$, β, I and L_M, where $N = \Delta\psi$ is the nutation in longitude.

Step 8 Next, in order to determine C', the position angle of the axis of rotation, determine Ω' from equations (14) and (15), and i from equation (13). Thus C' may be calculated from equations (18) and (19).

Step 9 Now use the ephemeris and equations (27)-(31) to determine the Euler angles ϕ_C, θ_C and ψ_C. This process includes the transformations from the system of the ephemeris (JPL DE403) to the true of date system equations (23)-(25).

Step 10 Then substitute ϕ_C for Ω, θ_C for I and $\psi_C + \phi_C - 180°$ for L_M and repeat steps (7), (8) and (9) resulting this time in the total librations l_T, b_T and C'_T.

Step 11 Lastly the physical librations, the differences between the total and the optical librations are given by equations (38)-(40).

4.2 Numerical Example

Calculate the quantities described in this Technical Note for 2011 June 01 at 0^{h}TT, approximately 21 hours before the instant of new Moon.

$$t = 2455713{\cdot}5000000 \quad \text{2011 June 01 } 0^{\mathrm{h}} \text{ TT}$$

Step 1: The apparent positions of the Moon and Sun have been calculated using methods described in Section B of *The Astronomical Almanac*.

Moon	α	=	57°364896851
	δ	=	22°200527037
	d	=	0·0026441632 au
Sun	α_S	=	68°564159796
	δ_S	=	21°975380381
	d_S	=	1·0139593548 au

Step 2: Nutations and the mean and true obliquity have been calculated using SOFA routines.

$N = \Delta\psi$	=	0°004500032
$\Delta\varepsilon$	=	−0°000366339
ε_m	=	23°437794624
$\varepsilon = \varepsilon_m + \Delta\varepsilon$	=	23°437428285